Asymptotic properties of estimators for continuous sampling designs with application to environmental surveys

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Résumé. Les inventaires forestiers nationaux sont basés sur des plans de sondage probabilistes. Il est courant de sélectionner aléatoirement un échantillon de points dans un continuum (le territoire étudié) puis de définir des supports de forme fixe (par exemple, des placettes ou des polygones) à partir de ces points pour réaliser l'enquête sur la population d'arbres ; voir par exemple Vidal et al. (2016) pour un aperçu des plans d'échantillonnage utilisés. Bien que le plan d'échantillonnage puisse être formalisé de plusieurs manières (voir par exemple Eriksson, 1995), l'approche de la population infinie (Stevens et Urqhart, 2000; Barabesi, 2003; Mandallaz, 2007; Grégoire et Valentine, 2007) est sans doute le dispositif le plus simple. L'inférence peut être effectuée directement à partir de la population échantillonnée, en utilisant la théorie de l'estimation de Horvitz-Thompson en population continue (Cordy, 1993), à la fois en termes d'estimation ponctuelle et d'estimation de variance (Chauvet et al., 2023). Afin de produire des estimateurs fiables, certaines propriétés importantes sont nécessaires à ces plans d'échantillonnage. L'estimateur de Horvitz-Thompson doit être consistant et asymptotiquement normalement distribué, et un estimateur de variance consistant doit également être disponible pour une estimation par intervalle. Ces propriétés ont été peu étudiées dans la littérature, à l'exception de Barebesi et Franceschi (2011) et Barabesi et al. (2012). Dans ce travail, nous obtenons ces propriétés de façon générale pour des plans de sondage en population continue, sous des conditions faibles.

Mots-clés. Consistance, Enquête environnementale, Inventaire forestier, Normalité asymptotique.

Abstract. National forest inventories are based on probabilistic sampling designs. It is common practice to randomly select a sample of points in a continuum (the territory under study) and then to define fixed-shape supports (e.g., plots or polygons) from these points to perform the survey on the field on the population of trees; see for example Vidal et al. (2016) for a worldwide overview of sampling designs used in forest inventories. Although the sampling design may be formalized in several manners (e.g., Eriksson, 1995), the infinite population approach (Stevens and Urqhart, 2000; Barabesi, 2003; Mandallaz, 2007; Gregoire and Valentine, 2007) is arguably the simplest device for inference. Inference may be performed directly from the sampled population, which is straightforward by using the theory of continuous Horvitz-Thompson (HT) estimation (Cordy, 1993), both in terms of point estimation and variance estimation (Chauvet et al., 2023). In order to produce reliable estimators, some important properties are needed for continuous sampling designs. The

HT-estimator should be consistent and asymptotically normally distributed, with consistent variance estimators for interval estimation. These properties have not been much considered in the literature, with the exception of Barebesi and Franceschi (2011) and Barabesi et al. (2012). In this work, we derive these properties for general continuous sampling designs, under mild conditions.

Keywords. Consistency, Environmental survey, Forest inventory, Asymptotic normality.

1 Continuous Horvitz-Thompson estimation

1.1 Notations

We are interested in a continuous universe $\mathcal{U}$ of surface area A, which is included in $\mathbb{R}^q$ for some $q \geq 1$. We consider some Q-vector of attributes $\boldsymbol{\rho} : \mathcal{U} \to \mathbb{R}^Q$ which is Lebesgue-integrable, and we wish to estimate the (integral) total

$$\boldsymbol{\tau_\rho} \quad = \quad \int_{\mathcal{U}} \boldsymbol{\rho}(x)\,\mathrm{d}x. \tag{1}$$

In case of forest inventories, $\mathcal{U}$ is an intermediary universe used to attain a population U of trees for which we want to estimate the total $\boldsymbol{Y} = \sum_{k \in U} \boldsymbol{y}_k$ of some Q-vector of interest $\boldsymbol{y}_k$. Making use of a non-negative link function $L(\cdot,\cdot) : \mathcal{U} \times U \to \mathbb{R}^+$ which depends on the form of the fixed-shape support, the attribute of interest $\boldsymbol{y}_k$ may be converted into the synthetic variable

$$\boldsymbol{\rho}(x) = \sum_{k \in U} \frac{L(x,k)\boldsymbol{y}_k}{M_{+k}} \text{ for } x \in \mathcal{U}, \tag{2}$$

with $M_{+k} = \int_{\mathcal{U}} L(x,k)\,\mathrm{d}x$ the measure of the sub-territory in $\mathcal{U}$ leading to the selection of unit k, see Stevens and Urquart (2000) and Chauvet et al. (2023). Provided that $M_{+k} > 0$ for any $k \in U$, the local variable $\boldsymbol{\rho}(x)$ is such $\int_{\mathcal{U}} \boldsymbol{y}(x)dx = \boldsymbol{Y}$, hence estimating the total $\boldsymbol{Y}$ is equivalent to estimate the integral in (1).

A random sample $S = \{x_1, \ldots, x_n\}$ of n locations is selected in $\mathcal{U}$ according to the joint probability density function (PDF) $f(x_1, \ldots, x_n)$. Following Cordy (1993), we define the inclusion density function for $x \in \mathcal{U}$ as

$$\pi(x) \quad = \quad \sum_{i=1}^{n} f_i(x), \tag{3}$$

where $f_i(\cdot)$ if the marginal PDF for the i^{th} draw, and we define the joint inclusion density function for $x, y \in \mathcal{U}^2$ as

$$\pi(x,y) \quad = \quad \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} f_{ij}(x,y), \tag{4}$$

where $f_{ij}(\cdot,\cdot)$ if the joint PDF for the draws i and j. The third order and fourth order inclusion density functions

$$\begin{aligned} \pi(x,y,z) &= \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\substack{i'=1\\ i'\neq i,j}}^{n} f_{iji'}(x,y,z), \qquad (5)\\ \text{and } \pi(x,y,z,t) &= \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\substack{i'=1\\ i'\neq i,j}}^{n}\sum_{\substack{j'=1\\ j'\neq i,j,i'}}^{n} f_{iji'j'}(x,y,z,t) \end{aligned}$$

are defined similarly, with $f_{iji'}(\cdot,\cdot,\cdot)$ the joint PDF for the draws i, j, j' and $f_{iji'j'}(\cdot,\cdot,\cdot,\cdot)$ the joint PDF for the draws i, j, j' and i'.

1.2 Horvitz-Thompson estimation

The Horvitz-Thompson (HT) estimator

$$\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi} = \sum_{x\in S}\frac{\boldsymbol{\rho}(x)}{\pi(x)} = \sum_{i=1}^{n}\frac{\boldsymbol{\rho}(x_i)}{\pi(x_i)} \qquad (6)$$

is design-unbiased for $\boldsymbol{\tau}_{\boldsymbol{\rho}}$, provided that $\pi(x)>0$ almost everywhere on $\mathcal{U}$ (Cordy, 1993). The covariance matrix of $\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi}$ is

$$\boldsymbol{V}_p(\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi}) = \int_{\mathcal{U}}\frac{\boldsymbol{\rho}(x)\boldsymbol{\rho}(x)^\top}{\pi(x)}\,\mathrm{d}x + \int_{\mathcal{U}}\int_{\mathcal{U}}\{\pi(x,y)-\pi(x)\pi(y)\}\frac{\boldsymbol{\rho}(x)\boldsymbol{\rho}(y)^\top}{\pi(x)\pi(y)}\,\mathrm{d}x\mathrm{d}y, \qquad (7)$$

and may be estimated by the Horvitz-Thompson variance estimator

$$\hat{\boldsymbol{V}}_{HT}(\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi}) = \sum_{i=1}^{n}\frac{\boldsymbol{\rho}(x_i)\boldsymbol{\rho}(x_i)^\top}{\pi(x_i)^2} + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\pi(x_i,x_j)-\pi(x_i)\pi(x_j)}{\pi(x_i,x_j)}\frac{\boldsymbol{\rho}(x_i)}{\pi(x_i)}\frac{\boldsymbol{\rho}(x_j)^\top}{\pi(x_j)}, \qquad (8)$$

and the Sen-Yates-Grundy variance estimator

$$\hat{\boldsymbol{V}}_{YG}(\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\pi(x_i)\pi(x_j)-\pi(x_i,x_j)}{\pi(x_i,x_j)}\left\{\frac{\boldsymbol{\rho}(x_i)}{\pi(x_i)}-\frac{\boldsymbol{\rho}(x_j)}{\pi(x_j)}\right\}\left\{\frac{\boldsymbol{\rho}(x_i)}{\pi(x_i)}-\frac{\boldsymbol{\rho}(x_j)}{\pi(x_j)}\right\}^\top. \qquad (9)$$

Cordy (1993) considered only the case $Q=1$, but his variance formulas are easily generalized to the multivariate case. For the fixed-size sampling designs that we consider in this paper, the two variance estimators are design-unbiased for $\boldsymbol{V}_p(\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi})$, provided that $\pi(x,y)>0$ almost everywhere on $\mathcal{U}^2$ (Cordy, 1993, Section 2).

1.3 Plug-in estimation

We also consider the case of smooth functions of totals, which is very important in practice. Suppose that we wish to estimate a parameter $\theta = g(\boldsymbol{\tau_\rho})$, with $g : \mathbb{R}^Q \to \mathbb{R}$ some known function. Let $\|\cdot\|$ denote the Euclidean norm. Let us denote

$$\boldsymbol{\mu_\rho} = \frac{\boldsymbol{\tau_\rho}}{A} \quad \text{and} \quad \hat{\mu}_{\boldsymbol{\rho}\pi} = \frac{\hat{\tau}_{\boldsymbol{\rho}\pi}}{A}. \tag{10}$$

We suppose that there exists some neighborhood $B_\eta(\boldsymbol{\mu_\rho}) = \{\mathbf{a} \in \mathbb{R}^q;\ \|\mathbf{a} - \boldsymbol{\mu_\rho}\| \leq \eta\}$ of $\boldsymbol{\mu_\rho}$, for some $\eta > 0$, such that $g(\cdot)$ is differentiable on $B_\eta(\boldsymbol{\mu_\rho})$, with $\boldsymbol{g}'(\boldsymbol{\mu_\rho}) \neq 0$. The plug-in estimator of θ, truncated to avoid impossible values, is

$$\hat{\theta}_\pi \;=\; \begin{cases} g(\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi}) & \text{if } \hat{\boldsymbol{\mu}}_{\boldsymbol{\rho}\pi} \in B_\eta(\boldsymbol{\mu_\rho}), \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

The linearization variance approximation for $\hat{\theta}_\pi$ is $V_p(\hat{\tau}_{l\pi})$, with

$$l(x) \;=\; \{\boldsymbol{g}'(\boldsymbol{\tau_\rho})\}^\top \boldsymbol{\rho}(x), \tag{12}$$

the linearized variable of θ, and where $\hat{\tau}_{l\pi}$ is obtained from (6) by replacing $\boldsymbol{\rho}(x)$ with $l(x)$. The truncated Sen-Yates-Grundy linearized variance estimator is obtained by replacing in equation (9) the variable $\boldsymbol{\rho}(x)$ with the estimated linearized variable

$$\hat{l}(x) \;=\; \{\boldsymbol{g}'(\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi})\}^\top \boldsymbol{\rho}(x), \tag{13}$$

which leads to

$$\hat{V}_{YG}(\hat{\theta}_\pi) = \begin{cases} \frac{1}{2}\sum_{i=1}^n \sum_{\substack{j=1 \\ j\neq i}}^n \frac{\pi(x_i)\pi(x_j) - \pi(x_i,x_j)}{\pi(x_i,x_j)} \left\{ \frac{\hat{l}(x_i)}{\pi(x_i)} - \frac{\hat{l}(x_j)}{\pi(x_j)} \right\}^2 & \text{if } \hat{\boldsymbol{\mu}}_{\boldsymbol{\rho}\pi} \in B_\eta(\boldsymbol{\mu_\rho}), \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

The Horvitz-Thompson linearized variance estimator

$$\hat{V}_{HT}(\hat{\theta}_\pi) = \begin{cases} \sum_{i=1}^n \left\{ \frac{\hat{l}(x_i)}{\pi(x_i)} \right\}^2 + \sum_{i=1}^n \sum_{\substack{j=1 \\ j\neq i}}^n \frac{\pi(x_i,x_j) - \pi(x_i)\pi(x_j)}{\pi(x_i,x_j)} \frac{\hat{l}(x_i)}{\pi(x_i)} \frac{\hat{l}(x_j)}{\pi(x_j)} & \text{if } \hat{\boldsymbol{\mu}}_{\boldsymbol{\rho}\pi} \in B_\eta(\boldsymbol{\mu_\rho}), \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

is obtained similarly.

1.4 General assumptions

The asymptotic framework for finite population sampling usually postulates that the population of interest belongs to a nested sequence of populations with increasing sizes (Isaki and Fuller, 1982), since if the sample size grows to infinity, the population size needs to grow accordingly. The asymptotic framework with continuous sampling designs is simpler, since there is no contradiction in having an increasing sample size inside an infinite universe (Mandallaz, 2007, p. 61). Therefore, we use the framework introduced in Mandallaz (1991), and simply suppose that the population $\mathcal{U}$ remains fixed, and that $n \to \infty$. This is also known as the Monte Carlo integration approach (Gregoire and Valentine, 2007, chapter 10).

We also consider the following assumptions:

H1: Some constant M_1 exists such that

$$\frac{1}{A}\int_{\mathcal{U}} \|\boldsymbol{\rho}(x)\|^4 \, \mathrm{d}x \quad \leq \quad M_1.$$

H2: Some constants $c_1, C_1 > 0$ exist such that for any $x, y \in \mathcal{U}^2$:

$$c_1 \frac{n^2}{A^2} \leq \pi(x,y) \leq C_1 \frac{n^2}{A^2}.$$

H3: Some constant C_2 exists such that

$$\sup_{x,y \in \mathcal{U}^2} |\Delta_2(x,y)| \leq C_2 \frac{n}{A^2}, \tag{16}$$

with $\Delta_2(x,y) = \pi(x,y) - \pi(x)\pi(y)$. Some constant C_3 exists such that

$$\sup_{x,y,z,t \in \mathcal{U}^4} |H_4(x,y,z,t)| \leq C_3 \frac{n^2}{A^4}, \tag{17}$$

where

$$\begin{aligned} H_4(x,y,z,t) &= \Delta_4(x,y,z,t) \\ &- \{\pi(x)\Delta_3(y,z,t) + \pi(y)\Delta_3(x,z,t) + \pi(z)\Delta_3(x,y,t) + \pi(t)\Delta_3(x,y,z)\} \\ &+ \{\pi(z)\pi(t)\Delta_2(x,y) + \pi(y)\pi(t)\Delta_2(x,z) + \pi(y)\pi(z)\Delta_2(x,t) \\ &\quad +\pi(x)\pi(t)\Delta_2(y,z) + \pi(x)\pi(z)\Delta_2(y,t) + \pi(x)\pi(y)\Delta_2(z,t)\} \end{aligned}$$

and with

$$\begin{aligned} \Delta_3(x,y,z) &= \pi(x,y,z) - \pi(x)\pi(y)\pi(z), \\ \Delta_4(x,y,z,t) &= \pi(x,y,z,t) - \pi(x)\pi(y)\pi(z)\pi(t). \end{aligned}$$

H4: $g(\cdot)$ is homogeneous of degree $\beta \geq 0$, i.e. $g(r\mathbf{a}) = r^\beta g(\mathbf{a})$ for any real $r > 0$ and any $\mathbf{a} \in \mathbb{R}^Q$.

H5: The differential $\boldsymbol{g}'$ is locally Lipschitz on $B_\eta(\mu_{\boldsymbol{\nu}})$, i.e. there exists some constant K such that for any Q-vectors $\mathbf{a}, \mathbf{b} \in B_\eta(\mu_{\boldsymbol{\nu}}) \times B_\eta(\mu_{\boldsymbol{\nu}})$, we have $\|\boldsymbol{g}'(\mathbf{b}) - \boldsymbol{g}'(\mathbf{a})\| \leq K\|\mathbf{b} - \mathbf{a}\|$.

It is assumed in (H1) that the vector of interest $\rho(x)$ has a finite moment of order 4, which holds in particular if all the components of $\rho(x)$ are bounded. This is a fairly weak assumption. Assumption (H2) is related to the joint inclusion density function, which is assumed to be bounded both above and below by the joint inclusion density function obtained under uniform random sampling. Assumption (H3) is related to the inclusion density functions of order 2 to 4. This is the equivalent of classical assumptions for finite population sampling, see for example Breidt and Opsomer (2000), assumptions (A6) and (A7). We show in Section 2 that assumptions (H2) and (H3) are satisfied in case of stratified sampling in $\mathcal{U}$ with a finite number of strata. The assumption (H4) was introduced in Deville (1999), and is satisfied with $\beta = 0$ for many parameters of interest like a ratio or a correlation coefficient, for example. The assumption (H5) is a mild regularity condition for $g(\cdot)$ in the neighborhood of $\boldsymbol{\mu_\rho}$. The condition of Lipschitz continuity is satisfied if $g(\cdot)$ is twice differentiable with a bounded Hessian matrix on $B_\eta(\boldsymbol{\mu_\rho})$.

1.5 Consistency of estimators

We first obtain in Proposition 1 the $\sqrt{n}$-consistency of the HT-estimators, and of the associated estimators of the covariance matrix.

Proposition 1. *Suppose that assumptions (H1)-(H3) hold. Then:*

$$\left\| V\left\{A^{-1}\left(\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi} - \boldsymbol{\tau}_{\boldsymbol{\rho}}\right)\right\}\right\| = O(n^{-1}), \tag{18}$$

$$E\left[\left\|A^{-2}n\left\{\hat{\boldsymbol{V}}_{HT}(\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi}) - \boldsymbol{V}_p(\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi})\right\}\right\|^2\right] = O(n^{-1}), \tag{19}$$

$$E\left[\left\|A^{-2}n\left\{\hat{\boldsymbol{V}}_{YG}(\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi}) - \boldsymbol{V}_p(\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi})\right\}\right\|^2\right] = O(n^{-1}), \tag{20}$$

where for a square matrix $\boldsymbol{B}$, $\|\boldsymbol{B}\|$ is the matrix norm induced by the Euclidean norm, a.k.a. the spectral norm.

Proposition 1 ensures that the HT-estimator is mean-square consistent, and that both the HT estimator and the YG estimator are also mean-square consistent for the covariance matrix, and may therefore be used for interval estimation. In the context of a forest inventory in a territory $\mathcal{U}$, Proposition 1 is of interest for both the estimation of surface attributes and of tree attributes. For a surface attribute, we may for example be interested in the domain $\mathcal{U}_d$ where forest is located. The corresponding area A_d may be written as the integral in (1), with $Q = 1$ and $\rho(x) = 1(x \in \mathcal{U}_d)$ the domain membership indicator. For a tree attribute, we may for example be interested in the total volume of wood $Y = \sum_{k\in U} y_k$, with U the population of trees located in $\mathcal{U}$, and y_k the volume of tree k. Making use of the synthetic variable in (2), Y may also we written as the integral in (1). Proposition 1 is therefore applicable for these two types of parameters.

Proposition 2. *Suppose that assumptions (H1)-(H5) hold. Then:*

$$E\left(A^{-2\beta}\left[(\hat{\theta}_\pi - \theta) - \{g'(\tau_{\boldsymbol{\rho}})\}^\top\{\hat{\tau}_{\boldsymbol{\rho}\pi} - \tau_{\boldsymbol{\rho}}\}\right]\right)^2 = O(n^{-2}), \tag{21}$$

$$E\left[A^{-2\beta}n\left|\hat{V}_{HT}(\hat{\theta}_\pi) - V_p(\hat{\tau}_{l\pi})\right|\right] = O(n^{-1/2}), \tag{22}$$

$$E\left[A^{-2\beta}n\left|\hat{V}_{YG}(\hat{\theta}_\pi) - V_p(\hat{\tau}_{l\pi})\right|\right] = O(n^{-1/2}). \tag{23}$$

Equation (21) in Proposition 2 implies that the bias of the truncated plug-in estimator is asymptotically negligible. Equation (21) also implies that the linearization variance approximation and the true variance of the truncated plug-in estimator are asymptotically the same.

The consistency in the L^1 norm of the HT and SYG variance estimators is established in equations (22) and (23), which is weaker than the mean-square consistency obtained in Proposition 1 for the estimators of the covariance matrix of $\hat{\tau}_{\boldsymbol{\rho}\pi}$. The mean square consistency can easily be established by using the same proof, and by strengthening (H1) to have a bounded moment of order 8. Anyway, the convergence in the L^1 norm of the variance estimators is sufficient to obtain the asymptotic normality of the studentized plug-in estimator in case of stratified sampling, see Section 2.

2 Stratified sampling

2.1 Notations

The population $\mathcal{U}$ is partitioned into H strata $\mathcal{U}^1, \ldots, \mathcal{U}^H$ with respective areas $A^1, \ldots, A^H$. Inside the stratum $\mathcal{U}^h$, a sample S^h is obtained by n^h independent selections according to the marginal PDF $g^h(\cdot)$, and

$$S = \bigcup_{h=1}^{H} S^h. \tag{24}$$

The integral total $\boldsymbol{\tau_\rho}$ in (1) may we rewritten as

$$\boldsymbol{\tau_\rho} = \sum_{h=1}^{H} \boldsymbol{\tau}_{\boldsymbol{\rho}}^h \quad \text{with} \quad \boldsymbol{\tau}_{\boldsymbol{\rho}}^h = \int_{\mathcal{U}^h} \boldsymbol{\rho}(x)\, \mathrm{d}x.$$

From equation (3), we have

$$\pi(x) = n^h g^h(x) \quad \text{for any} \quad x \in \mathcal{U}^h, \tag{25}$$

and the HT-estimator may be rewritten as

$$\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi} = \sum_{h=1}^{H} \hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi}^h \text{ with } \hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi}^h = \frac{1}{n^h} \sum_{x \in S^h} \frac{\boldsymbol{\rho}(x)}{g^h(x)}. \tag{26}$$

From equation (4), we have

$$\pi(x,y) = \begin{cases} n^h(n^h-1)g^h(x)g^h(y) & \text{if } x, y \in \mathcal{U}^h, \\ n^h n^{h'} g^h(x) g^{h'}(y) & \text{if } x \in \mathcal{U}^h \text{ and } y \in \mathcal{U}^{h'} \text{ with } h \neq h'. \end{cases} \tag{27}$$

In particular, note that we need $n^h \geq 2$ for any $h = 1, \ldots, H$ for $\pi(x, y)$ to be positive on $\mathcal{U}^2$. By using equation (5), the covariance matrix of $\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi}$ may be rewritten as

$$\begin{aligned} \boldsymbol{V}_p(\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi}) &= \sum_{h=1}^{H} \frac{\boldsymbol{\Sigma}^2_{\boldsymbol{\rho}h}}{n^h}, \\ \text{with } \boldsymbol{\Sigma}^2_{\boldsymbol{\rho}h} &= \int_{\mathcal{U}^h} g^h(x) \left\{ \frac{\boldsymbol{\rho}(x)}{g^h(x)} - \boldsymbol{\tau}^h_{\boldsymbol{\rho}} \right\} \left\{ \frac{\boldsymbol{\rho}(x)}{g^h(x)} - \boldsymbol{\tau}^h_{\boldsymbol{\rho}} \right\}^\top \mathrm{d}x. \end{aligned} \tag{28}$$

Suppose that $n^h \geq 2$ for any $h = 1, \ldots, H$. The Sen-Yates-Grundy variance estimator may be rewritten as

$$\begin{aligned} \hat{\boldsymbol{V}}_{YG}(\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi}) &= \sum_{h=1}^{H} \frac{\hat{\boldsymbol{\Sigma}}^2_{\boldsymbol{\rho}h}}{n^h}, \\ \text{with } \hat{\boldsymbol{\Sigma}}^2_{\boldsymbol{\rho}h} &= \frac{1}{n^h - 1} \sum_{x \in S^h} \left\{ \frac{\boldsymbol{\rho}(x)}{g^h(x)} - \hat{\boldsymbol{\tau}}^h_{\boldsymbol{\rho}\pi} \right\} \left\{ \frac{\boldsymbol{\rho}(x)}{g^h(x)} - \hat{\boldsymbol{\tau}}^h_{\boldsymbol{\rho}\pi} \right\}^\top, \end{aligned} \tag{29}$$

and the Horvitz-Thompson variance estimator is identical.

2.2 Assumptions

We consider the following assumptions for stratified sampling:

H1b: There exists some constant $M_2 > 0$ such that:

$$\sup_{h=1,\ldots,H} \frac{1}{A^h} \int_{\mathcal{U}_h} \|\boldsymbol{\rho}(x)\|^4 \, \mathrm{d}x \leq M_2.$$

H6: There exists some constants $c_5, C_5 > 0$ such that for any $h = 1, \ldots, H$ and any $x \in \mathcal{U}_h$:

$$c_5 \leq A^h g^h(x) \leq C_5.$$

H7: There exists some constants $c_6, C_6 > 0$ and $C_7, C_7 > 0$ such that for any $h = 1, \ldots, H$:

$$\begin{aligned} c_6 \frac{n}{H} &\leq n_h \leq C_6 \frac{n}{H}, \\ c_7 \frac{A}{H} &\leq A^h \leq C_7 \frac{A}{H}. \end{aligned}$$

It is assumed in (H1b) that the vector of interest $\boldsymbol{\rho}(x)$ has a finite moment of order 4 inside each stratum. It is assumed in (H6) that the probability density function is of the same order for all the points inside a given stratum. In particular, this assumption holds true in case of uniform sampling inside strata. It is assumed in (H7) that the sample size share is of same order for each stratum, and similarly that the surface area share is of same order for each stratum.

Theorem 1. *Suppose that S is selected by stratified sampling, with a finite number of strata H and $n^h \geq 2$ inside each stratum $\mathcal{U}^h$. Suppose that assumptions (H1b), (H6) and (H7) hold. Then assumptions (H1)-(H3) hold, and the conclusions of Proposition 1 hold true. Also, for any $h = 1, \ldots, H$:*

$$\sqrt{n^h}\left\{\hat{\boldsymbol{\tau}}^h_{\boldsymbol{\rho}\pi} - \boldsymbol{\tau}^h_{\boldsymbol{\rho}}\right\} \quad \longrightarrow_{\mathcal{L}} \quad \mathcal{N}\left(\mathbf{0}_Q, \boldsymbol{\Sigma}^2_{\boldsymbol{\rho}h}\right) \tag{30}$$

where $\longrightarrow_{\mathcal{L}}$ stands for the convergence in distribution, and with $\mathbf{0}_Q$ a null vector of size Q.

Suppose in addition that

H8: There exists some constants $q^1, \ldots, q^H \in]0,1[$ such that for any $h = 1, \ldots, H$:

$$\frac{n^h}{n} \quad \to \quad q^h.$$

Then

$$\sqrt{n}\left\{\hat{\boldsymbol{\tau}}_{\boldsymbol{\rho}\pi} - \boldsymbol{\tau}_{\boldsymbol{\rho}}\right\} \quad \longrightarrow_{\mathcal{L}} \quad \mathcal{N}\left(\mathbf{0}_Q, \sum_{h=1}^{H} \frac{\boldsymbol{\Sigma}^2_{\boldsymbol{\rho}h}}{p_h}\right). \tag{31}$$

If the number of strata H remains fixed as $n \to \infty$, the assumption (H8) is automatically satisfied for equal allocation $n^h = n/H$, proportional allocation $n^h = n\frac{A^h}{A}$, or Neyman allocation, for example.

By the mean-square consistency of $\hat{V}_{HT}(\rho\pi)$ and $\hat{V}_{YG}(\rho\pi)$ (see equations 19 and 20), we obtain under the assumptions of Theorem 1 that

$$\frac{\hat{V}_{HT}(\rho\pi)}{V_p(\rho\pi)} \to_{Pr} 1 \quad \text{and} \quad \frac{\hat{V}_{YG}(\rho\pi)}{V_p(\rho\pi)} \to_{Pr} 1, \tag{32}$$

where $\to_{Pr}$ stands for the convergence in probability. Therefore, an approximate two-sided $100(1-2\alpha)\%$ confidence interval for τ_ρ is given by

$$\left[\hat{\tau}_{\rho\pi} \pm u_{1-\alpha}\left\{\hat{V}_{HT}(\rho\pi)\right\}^{0.5}\right] \quad \text{or} \quad \left[\hat{\tau}_{\rho\pi} \pm u_{1-\alpha}\left\{\hat{V}_{YG}(\rho\pi)\right\}^{0.5}\right] \tag{33}$$

with $u_{1-\alpha}$ the quantile of order $1-\alpha$ of the standard normal distribution.

Theorem 2. *Suppose that S is selected by stratified sampling, with a finite number of strata H and $n^h \geq 2$ inside each stratum $\mathcal{U}^h$. Suppose that assumptions (H1b) and (H4)-(H7) hold. Then the conclusions of Proposition 2 hold true. If in addition assumption (H9) holds, then:*

$$\sqrt{n}A^{-\beta}\left\{\hat{\theta}_\pi - \theta\right\} \quad \longrightarrow_{\mathcal{L}} \quad \mathcal{N}\left(\mathbf{0}_Q, \frac{1}{A^2}\sum_{h=1}^{H} \frac{\{g'(\mu_{\boldsymbol{\rho}})\}^\top \boldsymbol{\Sigma}^2_{\boldsymbol{\rho}h}\{g'(\mu_{\boldsymbol{\rho}})\}}{p_h}\right). \tag{34}$$

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