Election Robustness Index

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Abstract

Voting rules like majority judgement, range voting or approval voting are based on evaluations. In this context, we propose a new index giving the "Robustness of the election" of the winner of the voting process. This index is based on properties of depth contours.
Keywords : Depth functions, Depth contours, Evaluation based voting rules, Election Robustness Index.

1 Introduction

Representative democracy is widely present in developed countries. It heavily relies on elections during which representatives of the people (mayors, senators, deputies, president, etc.) are elected. The question of the legitimacy of elected officials is regularly addressed. In this work, we propose to provide a tool for assessing this legitimacy. This Election Robustness Index is developed within the framework of elections based on evaluations, specifically in the context of deepest voting linked to weighted L^p depth functions (see Zuo [7]). These voting processes encompass the methods of majority judgment, range voting, and approval voting, the three most common voting techniques based on evaluations.

Initially, we recall what deepest voting entails, then we introduce the Election Robustness Index for the winning candidate, as well as associated indices for unsuccessful candidates. Lastly, we examine a set of properties verified by the index and we study a case using simulated data.

2 Weighted L^p deepest voting

Deepest Voting (see Aubin et al. [1]) is a family of social decision functions based on evaluations. Let consider in the following that we have n voters and d candidates, and each voter give a grade to each candidate. Without loss of generality, we suppose that these grades are in $[0;1]$. Each voter can then be seen as a point in $[0;1]^d$ whose components are the grades for each candidate. The set of all the voters' grades is then a point cloud. The key idea of Deepest Voting is to consider the grades of the *most central* voter of the cloud. This innermost (possibly imaginary) voter can be seen as the most representative of all the voters of the cloud, so that his preferences should meet the largest possible consensus among the other voters. The associated social decision function simply gives the grades of this innermost voter as output.

Quoting Liu et al. [5]: "associated with a given distribution F on $\mathbb{R}^d$, a depth function is designed to provide a F-based center-outward ordering (and thus a ranking) of points x in $\mathbb{R}^d$. High depth corresponds to *centrality*, low depth to *outlyingness*". In other words, a depth function takes high (positive) values at the middle of a point cloud and vanishes out of it (see Zuo and Serfling [8] for a rigorous definition of a depth function). The intuitive key idea of Deepest Voting is enriched by a large choice of depth functions.

Let us introduce in particular the weighted L^p depths. Consider a distribution of n points $\Phi_n := \big(\Phi(\cdot,1),\dots,\Phi(\cdot,n)\big)$ in $\mathbb{R}^d$. The weigthed L^p depth at a point $x \in \mathbb{R}^d$ is defined by Zuo [7] as:

$$wL^pD(x) = \frac{1}{1+\frac{1}{n}\sum_{j=1}^{n}\omega\big(\|\Phi_n(\cdot,j)-x\|_p\big)}$$

where $p>0$, ω is a non-decreasing and continuous function on $[0,\infty)$ with $\omega(\infty-)=\infty$ and $\|x-x'\|_p = \left(\sum_{i=1}^{d}|x_i-x'_i|^p\right)^{1/p}$. If $\omega : x \to x^p$, then for $x=(x_1,\dots,x_d)$,

$$wL^pD(x) = \frac{1}{1+\frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{d}|\Phi_n(i,j)-x_i|^p}.$$

Not considering tie-breaking procedure, Majority Judgment and Range Voting (and its particular case, the approval voting) then correspond respectively to wL^1 Deepest Voting and wL^2 Deepest Voting. These are the three most famous voting processes based on evaluations.

The approval voting (see Brams and Fishburn [3] for a complete study) is maybe the most famous of these methods: each voter evaluates candidates on a scale of 2 gradings, which is the simplest possible scale. The voter gives a 1 if the candidate is acceptable, else a 0. The voter can then votes for several candidates (even all of them), or none of them accordingly to his convictions. Note that this method is very simple to apply in practice.

The two other methods are based on more nuanced classes of grading, which can be continuous or on a discrete scale. With the range voting proposed by Smith [6], the winner is the candidate with the highest average grade. With the majority judgment introduced by Balinski and Laraki [2], the winner is the candidate with the highest median grade. Note that the tie-break situation is taken into account for example in Fabre [4].

We visualise on figure 1 the example of the wL^2 depth function given in table 1.

voter	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
c_1	0.30	0.10	0.70	0.70	0.65	0.70	0.20	0.15	0.20
c_2	0.20	0.90	0.10	0.16	0.10	0.14	0.36	0.30	0.34

Table 1: Example of a distribution Φ_9 of grades given by 9 voters on 2 candidates.

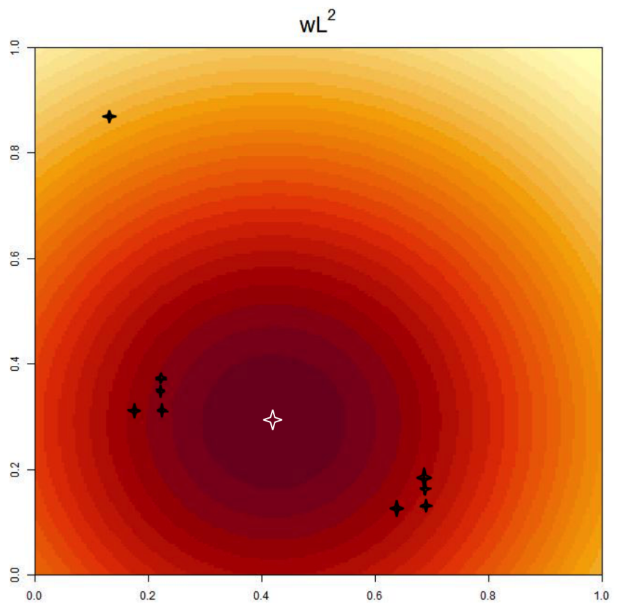

Figure 1: Example of wL^2 depth function on a data set. Horizontal axes give the grade for candidate c_1 and vertical axes for candidate c_2. Each black cross corresponds to a voter. The white cross corresponds to the deepest point.

3 Election Robustness Index for weighted L^p deepest voting

The notion of $\alpha-$trimmed region, proposed in Zuo and Serfling [8], is introduced to define an Election Robustness Index.

Definition 1. *For a given depth function D and a discrete distribution Φ_n and for $\alpha > 0$, we call*

$$D^\alpha(\Phi_n) := \left\{x \in [0;1]^d : D(x;\Phi_n) \geq \alpha\right\}$$

the corresponding $\alpha-$trimmed region.

Zuo and Serfling say that "The $\alpha-$trimmed region of a depth D (exhibits the structure of underlying multivariate distribution and) reveals the shape of multivariate datasets".

Note that for $\alpha_1 \geq \alpha_2$, $D^{\alpha_1}(\Phi_n) \subseteq D^{\alpha_2}(\Phi_n)$. It means that the $\alpha-$trimmed regions are nested. On figure 1, the $\alpha-$trimmed regions are the nested areas of same color. The deepest point for the wL^2 depth has coordinates (0.41, 0.29), inside the smallest $\alpha-$trimmed region, corresponding to the means componentwise.

Let denote by D_p^α the $\alpha-$trimmed regions for a wL^p depth. Zuo and Serfling [8] show that D_p^α are compact for a distribution Φ continuous.

Let introduce the notion of preferential area $\mathcal{A}_i$ for the candidate c_i as follows :

$$\mathcal{A}_i := \left\{x = (x_1, \ldots, x_d) \in [0;1]^d : \forall j \neq i,\ x_i \geq x_j\right\}.$$

Any set of preferences x included in $\mathcal{A}_i$ has in common that the prefered candidate is c_i. Moreover, let consider the frontier of $\mathcal{A}_i$, denoted $\partial\mathcal{A}_i$, and defined as the set of points $x = (x_1, \ldots, x_d) \in [0;1]^d$ such that $x \in \mathcal{A}_i$ and it exists $j \neq i$ with $x_j = x_i$.

For example, in the case of 2 candidates, $\mathcal{A}_1$ is the set of $x = (x_1, x_2)$ such that $x_1 \geq x_2$. $\mathcal{A}_1$ is the area under its frontier $\partial\mathcal{A}_1 = \left\{x = (x_1, x_2) : x_1 = x_2\right\}$ materialized by the red line in Figure 2.

The Election Robustness Index represents the "centrality" of the deepest point inside the preferential area of the winner. It represents the "distance" of the deepest point from the preferential area of a loser.

Definition 2. *Given the preferences Φ_n and the deepest point associated to the wL^p deepest voting x_p^*. Let denote $\partial\mathcal{A}_i$ the frontier of $\mathcal{A}_i$ and $\mathbb{1}_{\{x_p^* \in \mathcal{A}_i\}}$ the indicator function which gives 1 if $x_p^* \in \mathcal{A}_i$ and 0 elsewhere. Then the Election Robustness Index of candidate c_i denoted $ERI_p(i)$ is given by :*

$$ERI_p(i) = \left(2\mathbb{1}_{\{x_p^* \in \mathcal{A}_i\}} - 1\right)\left(1 - \frac{\max_{x \in \partial\mathcal{A}_i} D_p(x)}{D_p(x_p^*)}\right)$$

Note that only one candidate is associated to a positive ERI_p (the winner of the wL^p deepest voting procedure). All the others candidates have a negative ERI_p. The ERI_p has always an absolute value lower than 1. The greater the ERI_p is, the more robust the election is. In case of ex-aequos, their respective ERI_p are null.

Proposition 1. *The previous properties are summarized in the following points :*

- *If $\exists!\ \ i_0$ such that $x_p^* \in \mathcal{A}_{i_0}$, then $ERI_p(i_0) > 0$ and $\forall i \neq i_0, ERI_p(i) < 0$ and $\max_{i \neq i_0} ERI_p(i) = -ERI_p(i_0)$.*
- *If $\exists i_0 \neq i_1$ such that $x_p^* \in \mathcal{A}_{i_0} \cap \mathcal{A}_{i_1}$, then $ERI_p(i_0) = ERI_p(i_1) = 0$.*
- *If all elements of Φ_n are identical and included in the interior of $\mathcal{A}_{i_0}$, then $ERI_p(i_0) = 1$ and $\forall i \neq i_0, ERI_p(i) = -1$.*
- *For all i, $-1 \leq ERI_p(i) \leq 1$ and the winner is the only candidate with positive ERI_p.*

Next proposition is dedicated to a close form of the ERI_2 in the case of 2 candidates.

Proposition 2. *Let consider a matrix* Φ_n *of n grades on 2 candidates and the* wL^2 *depth then, if* $d := \left(\frac{\overline{\Phi_n(2,\cdot)}-\overline{\Phi_n(1,\cdot)}}{2}\right)^2$*, then, if we assume that candidate 1 is the winner,*

$$ERI_2(1) = \frac{2d}{1 + var\big(\Phi_n(1,\cdot)\big) + var\big(\Phi_n(2,\cdot)\big) + 2d} = -ERI_2(2).$$

The application of the previous proposition to Table 1 leads to $d = 4.10^{-3}$ and $ERI_2(1) = 6.10^{-3}$. The Election Robustness Index is very small which illustrates the narrow victory of candidate 1. Moreover, Figure 2 visualises the largest deepest contour included in $\mathcal{A}_1$.

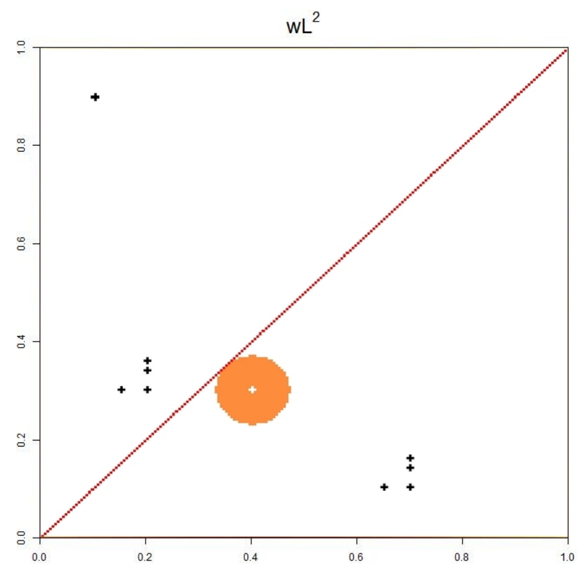

Figure 2: Example of the largest $\alpha-$trimmed region associated to a wL^2 depth function included in $\mathcal{A}_1$ in orange. Horizontal axes give the grade for candidate c_1 and vertical axes for candidate c_2. Each black cross corresponds to a voter, the white cross in the middle of the orange area is the deepest point.

Note to conclude that the choice of p in the wL^p depth is crucial. For example, p=1 or a p large enough would lead to the election of candidate 2 in Table 1.

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