Random coefficients regressions and radon transform inversion using mollification

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Résumé. Cet article considère le modèle de coefficients aléatoires et propose une approche par mollification pour régulariser la mal-position induite par l'inversion de la transformée de Radon. Nous analysons les propriétés asymptotiques et en déduisons les taux de convergence. Nous comparons notre estimateur aux méthodes classiques existantes à l'aide de simulations.

Mots-clés. Modèle à coefficients aléatoires, Problèmes inverses, Transformée de Radon, Mollification

Abstract. This paper considers the random coefficients model setting and proposes a mollification approach to regularize the ill-posedness induced by the inversion of the radon transform. We analyse asymptotic properties and derive rates of convergence. We compare our estimator with classical existing methods using simulations.

Keywords. Random coefficient models, inverse problems, radon transform, mollification

1 Model

Linearity in a causal relationship between a dependent variable and a set of regressors is a common assumption throughout economics. In this paper we consider the case when the coefficients in this relationship are random and distributed independently from the regressors. Our aim is to identify and estimate the distribution of the coefficients nonparametrically. The analysis of such model involves the inversion of the Radon transform, which is an ill-posed inverse problem to solve, see Beran and Hall (1992), Beran et al. (1996), Holderlein et al. (2010) for a presentation of this model.

Following Holderlein et al. (2010), we consider the following regression model

$$Y_i = b_i^T X_i,$$

where Y_i is an observed continuously distributed random scalar, X_i denotes an observed random d-vector of individual specific regressors and b_i is an unobserved random d-vector of individual coefficients. We assume that $Y \in \mathscr{R}$, $X \in \mathscr{R}^d$ and b is independent from X.

We first consider the following transformation of the random variables: $(Y_i, X_i) \mapsto (U_i, S_i)$, $i = 1, ..., n$ where

$$S_i = \|X_i\|^{-1} X_i \in \mathscr{S}_{d-1}, \quad U_i = \|X_i\|^{-1} Y_i \in \mathscr{R},$$

and the unit sphere in $\mathscr{R}^d$ is denoted by $\mathscr{S}_{d-1} = \{z \in \mathscr{R}^d : \|z\| = 1\}$. Then, the model becomes:

$$U_i = b_i^T S_u.$$

Second, we consider the $d-1$ hyperplanes defined by a direction vector $s \in \mathscr{S}_{d-1}$ and a distance from the origin $u \in \mathscr{R}$:

$$P_{s,u} = \{z \in \mathscr{R}^d : \ z^T s = u\}.$$

The Radon transform of a function $f : \mathscr{R}^d \to \mathscr{R}$ is defined as

$$(Rf)(s, u) = \int_{P_{s,u}} f.$$

The conditional density of U given S is given by the Radon transform of the density f_b:

$$f_{U|S}(u|s) = Rf_b(s, u).$$

To solve this ill-posed inverse problem and recover the function f_b, Holderlein et al. (2010) propose to use a regularized inverse A_h of the Radon transform:

$$(A_h g)(z) = \int_{S_{d-1}} \int_{-\infty}^{\infty} K_h(s^T z - u) g(s, u) du d\mu(s),$$

where μ is the Lebesgue measure on S_{d-1} and K is a smoothing kernel. A regularized solution for the density f_b could then be defined as:

$$f_b(z) = (A_h f_{U|S)}(z) = \int_{S_{d-1}} \int_{-\infty}^{\infty} K_h(s^T z - u) f_{U|S}(u|s) du d\mu(s).$$

2 Mollification

In this paper, we propose a different method to regularize the deconvolution problem, which uses a regularization principle introduced in the deterministic setting, and has been applied in several fields of signal and image processing such as deconvolution of images in astronomy, computerized tomography, as in as in Alibaud et al. (2009). We refer to it as the regularization by mollification, or merely as the mollification, Bonnefond and Maréchal (2009) for a general presentation of the method.

For general ill-posed equations $Rf_b = f_{U|S}$, the variational setting consists in defining a solution *via* an optimization problem the form

$$\text{Minimize} \quad \mathscr{D}(Rf_b, f_{U|S}) + \mathscr{R}(f_b).$$

Here, the first term (referred to as the *data fidelity term*) invites the solution to somehow respect the model while the second term (referred to as the *regularization term*) aims at stabilizing the solution with respect to perturbations of the data $f_{U|S}$.

We call *mollification* the variational counterpart of approximate inverses. The same target, namely

$$C_\beta f_b := \varphi_\beta * f_b,$$

is sought for, but we now use the variational setting. From obvious heuristics, the undesired component of f_b is $(I - C_\beta) f_b$, which suggests that the regularization term should merely be $\mathscr{R}(f_b) := \|(I - C_\beta) f_b\|^2$. As for the data fidelity term, we observe that the target $\varphi_\beta * f_b^\dagger$ satisfies the equation

$$R(\varphi_\beta * f_b) = \varphi_\beta * f_{U|S} = C_\beta f_{U|S},$$

since the radon transform possesses similar associativity and commutativity properties as the convolution. This suggests to take a data fidelity term of the form

$$\mathscr{D}(Rf, f_{U|S}) = \left\| Rf - C_\beta f_{U|S} \right\|^2.$$

In summary, the original mollification approach to the radon transform problem consists in defining the reconstructed density as the (unique) solution to the minimization problem

$$f_{\mathrm{MO},\beta} := \operatorname{argmin}_f \left[\left\| C_\beta f_{U|S} - Rf \right\|^2 + \left\| (I - C_\beta) f \right\|^2 \right]. \tag{1}$$

We will also consider the modified version of the mollification, as in Hohage et al. (2022):

$$f_{\mathrm{MM},\beta} := \operatorname{argmin}_f \left[\left\| f_{U|S} - Rf \right\|^2 + \left\| (I - C_\beta) f \right\|^2 \right].$$

By the first order optimality conditions we have the explicit formula

$$f_{\mathrm{MO},\beta} = \left(R^* R + (I - C_\beta)^* (I - C_\beta) \right)^{-1} R^* C_\beta f_{U|S}.$$

We note that $R^* R$ is positive definite since R is injective, so that the operator $R^* R + (I - C_\beta)^* (I - C_\beta)$ is also positive definite.

Using mollification approach, we build a regularized estimator of the radon transform inversion that is proved to be convergence with optimal rates of convergence. We also compare the finite sample properties of our estimator with other estimators found in the literature.

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