Detecting the change points in a nonlinear time series models for weakly dependent observations

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Résumé. Cet article étudie la détection de rupture d'une classe de modèles paramétriques conditionnels hétéroscédastiques non linéaires autorégressifs (CHARN). Les estimateurs conditionnels des moindres carrés (CLS) des paramètres sont définis et se révèlent cohérents. Un estimateur de la détection de rupture est défini. Sa consistance et sa distribution limite sont étudiées en détail.

Mots-clés. ruptures, modèles CHARN, moindres carrés conditionnels, mélange, tests

Abstract. This paper studies change-point detection of a class of parametric conditional heteroscedastic autoregressive nonlinear (CHARN) models. The conditional least-squares (CLS) estimators of the parameters are defined and are proved to be consistent. An estimator of the change-point location is defined. Its consistency and its limiting distribution are studied in detail.

Keywords. change-points, CHARN models, conditional least-squares, mixing, tests

1 Introduction

Detecting jumps in a series of real numbers and determining their number and locations is known in statistics as a change-point problem. This is usually solved by testing for the stationarity of the series and estimating the change locations when the null hypothesis of stationarity is rejected.

The study of conditional variations in financial and economic data receives particular attention as a result of its interest in hedging strategies and risk management.

The literature on change-points is vast. A popular alternative to using the likelihood ratio test was employed in Hinkley (1970,1972). Yao and Davis (1986) reviewed the asymptotic behavior of likelihood ratio statistics for testing a change in the mean in a series of iid Gaussian random variables. Csörgő and Horváth (1987) came up with statistics based on linear rank statistical processes with quantum scores. Chen and Gupta (1999) looked at detection tests and change-point estimation methods for models based on the normal distribution. The contribution of Horváth and Steinebach (2000) is related to the change in mean and variance. Antoch and Hušková (2001) proposed permutation tests for the location and scale parameters of a law. Hušková and Meintanis (2006) developed a change-point test using the empirical characteristic functions. Zou et al. (2007) proposed several CUSUM approaches. Gombay (2008) proposed tests for change detection in the mean, variance, and autoregressive parameters of a p-order autoregressive model.

There are several types of changes depending on the temporal behavior of the series studied. The usual ones are abrupt change, gradual change, and intermittent change. In this paper, we focus on abrupt change in the conditional variance of *off-line* data issue from a class of CHARN models (see Härdle and Tsybakov (1997) and Härdle et al. (1998)). These models are of the most famous and significant ones in finance, which include many financial time series models. We suggest a hybrid estimation procedure, which combines CLS and non-parametric methods to estimate the change location. Indeed, conditional least-squares estimators own a computational advantage and require no knowledge of the innovation process.

The rest of the paper is organized as follows. Section 2 presents the class of models studied, the notation, the main assumptions and the main result on the CLS estimators of the parameters. Section 3 presents the change-point test and the change location LS estimation. The asymptotic distribution of the test statistic under the null hypothesis is investigated. The consistency rates are obtained for the change location estimator and its limit distribution is derived.

2 Model and Assumptions

We place ourselves in the framework where the observations at hand are assumed to be issued from the following CHARN (p,p) model :

$$X_t = m\left(\rho; Z_{t-1}\right) + \sigma\left(\theta; Z_{t-1}\right)\varepsilon_t,\ t \in \mathbb{Z}, \tag{1}$$

where $p \in \mathbb{N}^* \bigcup \{\infty\}$; $m(\cdot)$ and $\sigma(\cdot)$ are two real-valued functions of known forms depending on unknown parameters ρ and θ, respectively ; for all $t \in \mathbb{Z}$, $Z_{t-1} = (X_{t-1}, X_{t-2}, \ldots, X_{t-p})^{\intercal}$; $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a sequence of stationary random variables with $\mathbb{E}\left(\varepsilon_t \mid Z_{t-1}\right) = 0$ and $\mathbb{V}\left(\varepsilon_t \mid Z_{t-1}\right) = 1$ such that ε_t is independent of the σ-algebra $\mathcal{F}_{t-1} = \sigma\left(Z_k,\ k < t\right)$. The case $p = \infty$ is treated in Bardet and Kengne (2014) where the stationarity and the ergodicity of the process $(X_t)_{t\in\mathbb{Z}}$ is studied. Although we restrict to $p < \infty$, all the results stated here also hold for $p = \infty$.

Let $\psi = (\rho^{\intercal}, \theta^{\intercal})^{\intercal} \in \Psi = \mathrm{int}(\Theta) \times \mathrm{int}(\widetilde{\Theta}) \subset \mathbb{R}^r \times \mathbb{R}^l$, the vector of the parameters of the model (1) and $\psi_0 = (\rho_0^{\intercal}, \theta_0^{\intercal})^{\intercal}$ the true parameter vector. Denote by $\|M\|$ an appropriate norm of a vector or a matrix M. We assume that all the random variables in the whole text are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
We make the following assumptions :

(A_1) The common fourth order moment of the ε_t is finite.

(A_2)

- The function $m(\cdot)$ is twice continuously differentiable, a.e., with respect to ρ in some neighborhood $\mathcal{B}_1$ of ρ_0.
- The function $\sigma(.)$ is twice continuously differentiable, a.e., with respect to θ in some neighborhood $\mathcal{B}_2$ of θ_0.

- There exists a positive function ω such that $\mathbb{E}\left(\omega^4\left(Z_0\right)\right)<\infty$, and

$$\max\left\{\sup_{\rho\in\mathrm{int}(\Theta)}|m(\rho;z)|\,,\ \sup_{\rho\in\mathrm{int}(\Theta)}\|\partial_\rho m(\rho;z)\|\,,\ \sup_{\rho\in\mathrm{int}(\Theta)}\left\|\partial^2_{\rho^2}m(\rho;z)\right\|\right\}\leqslant\omega(z)$$

$$\max\left\{\sup_{\theta\in\mathrm{int}(\widetilde{\Theta})}|\sigma(\theta;z)|\,,\ \sup_{\theta\in\mathrm{int}(\widetilde{\Theta})}\|\partial_\theta \sigma(\theta;z)\|\,,\ \sup_{\theta\in\mathrm{int}(\widetilde{\Theta})}\left\|\partial^2_{\theta^2}\sigma(\theta;z)\right\|\right\}\leqslant\omega(z).$$

(A_3) There exists a positive function β such that $\mathbb{E}\left(\beta^4\left(Z_0\right)\right)<\infty$, and for all $\rho_1,\ \rho_2\in\mathrm{int}(\Theta)$, and $\theta_1,\ \theta_2\in\mathrm{int}(\widetilde{\Theta})$,

$$\begin{aligned}\max\Big\{&\left|m\left(\rho_1;z\right)-m\left(\rho_2;z\right)\right|,\left\|\partial_\rho m\left(\rho_1;z\right)-\partial_\rho m\left(\rho_2;z\right)\right\|,\\&\left\|\partial^2_{\rho^2}m\left(\rho_1;z\right)-\partial^2_{\rho^2}m\left(\rho_2;z\right)\right\|,\left|\sigma\left(\theta_1;z\right)-\sigma\left(\theta_2;z\right)\right|,\\&\left\|\partial_\theta\sigma\left(\theta_1;z\right)-\partial_\theta\sigma\left(\theta_2;z\right)\right\|,\left\|\partial^2_{\theta^2}\sigma\left(\theta_1;z\right)-\partial^2_{\theta^2}\sigma\left(\theta_2;z\right)\right\|\Big\}\\&\leqslant\beta(z)\min\left\{\left\|\rho_1-\rho_2\right\|_2,\left\|\theta_1-\theta_2\right\|_2\right\}.\end{aligned}$$

(A_4) The sequence $(\varepsilon_t)_{t\in\mathbb{Z}}$ is stationary and satisfies either of the following two conditions :
- α-mixing with mixing coefficient satisfying $\sum_{n\geq1}\left[\alpha(n)\right]^{\delta/(2+\delta)}<\infty$ and $\mathbb{E}|\varepsilon_0|^{2+\delta}<\infty$ for some $\delta>0$;
- ϕ-mixing with mixing coefficient satisfying $\sum_{n\geqslant1}\left[\phi(n)\right]^{1/2}<\infty$ and $\mathbb{E}\left|\varepsilon_0\right|^{4+\delta}<\infty$ for some $\delta>0$.

The conditional mean and the conditional variance of X_t are given, respectively, by

$$\mathbb{E}\left(X_t\mid\mathcal{F}_{t-1}\right)=m\left(\rho;Z_{t-1}\right)\quad\text{and}\quad\mathbb{V}\left(X_t\mid\mathcal{F}_{t-1}\right)=\sigma^2\left(\theta;Z_{t-1}\right).$$

From these, one has that for all $z\in\mathbb{R}^p$,

$$\mathbb{E}\left(X_1\mid Z_0=z\right)=m\left(\rho;z\right)\quad\text{and}\quad\mathbb{E}\left(\left(X_1-m\left(\rho;Z_0\right)\right)^2\mid Z_0=z\right)=\sigma^2\left(\theta;z\right).$$

Therefore, for any bounded measurable functions $g(\cdot)$ and $k(\cdot)$, we have

$$\mathbb{E}\left\{\left[X_1-m\left(\rho;Z_0\right)\right]g(Z_0)\right\}=0\quad\text{and}\quad\mathbb{E}\left(\left\{\left[X_1-m\left(\rho;Z_0\right)\right]^2-\sigma^2\left(\theta;Z_0\right)\right\}k(Z_0)\right)=0.$$

Without a loss of generality, in the following we take, for all $z\in\mathbb{R}^p$, $g(z)=k(z)=1$. Now, given $X_{-p+1},\ldots,X_{-1},X_0,X_1,\ldots,X_n$ with $n\gg p$, we let $\mathbb{X}_n=(X_{-p+1},\ldots,X_{-1},X_0,X_1,\ldots,X_n)$ and consider the sequences of random functions

$$\begin{aligned}Q_n(\rho)=Q_n(\rho;\mathbb{X}_n)&=\sum_{t=1}^n\left(X_t-\mathbb{E}\left(X_t\mid\mathcal{F}_{t-1}\right)\right)^2=\sum_{t=1}^n\left(X_t-m\left(\rho;Z_{t-1}\right)\right)^2\\S_n(\rho,\theta)=S_n(\rho,\theta;\mathbb{X}_n)&=\sum_{t=1}^n\left(\left(X_t-m\left(\rho;Z_{t-1}\right)\right)^2-\sigma^2\left(\theta;Z_{t-1}\right)\right)^2.\end{aligned}$$

We have the following theorem :

Theorem 1 *Under assumptions* (A_1)-(A_3), *there exists a sequence of estimators* $\widehat{\psi}_n = \left(\widehat{\rho}_n^\intercal, \widehat{\theta}_n^\intercal\right)^\intercal$ *such that* $\widehat{\psi}_n \longrightarrow \psi_0$ *almost surely, and for any* $\epsilon > 0$, *there exists an event* E *with* $\mathbb{P}(E) > 1-\epsilon$, *and a non-negative integer* n_0 *such that on* E, *for* $n > n_0$,

- $\dfrac{\partial Q_n}{\partial \rho}(\widehat{\rho}_n; \mathbb{X}_n) = 0$ *and* $Q_n(\rho; \mathbb{X}_n)$ *attains a relative minimum at* $\rho = \widehat{\rho}_n$ *;*
- *assuming* $\widehat{\rho}_n$ *fixed,* $\dfrac{\partial S_n}{\partial \theta}\left(\widehat{\psi}_n; \mathbb{X}_n\right) = 0$ *and* $S_n\left((\widehat{\rho}_n, \theta); \mathbb{X}_n\right)$ *attains a relative minimum at* $\theta = \widehat{\theta}_n$.

Sketch of the proof. This result is an extension of Ngatchou-Wandji (2008) to the case $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a mixing martingale difference. $\square$

3 Change-Point Study

3.1 Change-Point Test and Change Location Estimation

We essentially use the techniques of Bai (1994), who studied the estimation of the shift in the mean of a linear process by a LS method. We first consider the model (1) for known ρ, and $\sigma(\theta; Z_{t-1}) = \underline{\theta}\delta_0(Z_{t-1})$, for some known positive real-valued function $\delta_0(\cdot)$ defined on $\mathbb{R}^p$ and for an unknown positive real number $\underline{\theta}$. We wish to test

$$H_0 : \underline{\theta} = \vartheta_1 = \vartheta_2 \text{ over } t \leqslant n$$

against

$$H_1 : \underline{\theta} = \begin{cases} \vartheta_1, & t = 1, \dots, t^* \\ \vartheta_2, & t = t^*+1, \dots, n. \end{cases} \qquad (\vartheta_1 \neq \vartheta_2)$$

where $\vartheta_1,\ \vartheta_2$ and t^* are unknown parameters.
If we put

$$\sigma^2(\theta; Z_{t-1}) = \delta(Z_{t-1}) = \underline{\theta}\delta_0(Z_{t-1}) = \begin{cases} \vartheta_1\delta_0(Z_{t-1}), & t = 1, \dots, t^* \\ \vartheta_2\delta_0(Z_{t-1}), & t = t^*+1, \dots, n. \end{cases} \qquad (\vartheta_1 \neq \vartheta_2)$$

and if we are also interested in estimating $\vartheta_1,\ \vartheta_2$ and the change location t^*, when H_0 is rejected. It is assumed that $t^* = [n\tau]$ for some $\tau \in (0,1)$, with $[x]$ standing for the integer part of any real number x. From (1), one can easily check that

$$\left(X_t - m(\rho; Z_{t-1})\right)^2 = \delta^2(Z_{t-1}) + \delta^2(Z_{t-1})\left(\varepsilon_t^2 - 1\right),\ t \in \mathbb{Z} \qquad (2)$$

and

$$\frac{\left(X_t - m(\rho; Z_{t-1})\right)^2}{\delta_0^2(Z_{t-1})} = \begin{cases} \vartheta_1^2\varepsilon_t^2, & t = 1, \dots, t^* \\ \vartheta_2^2\varepsilon_t^2, & t = t^*+1, \dots, n. \end{cases} \qquad (\vartheta_1 \neq \vartheta_2)$$

from which we define the LS estimator $\widehat{t^*}$ of t^* as follows :

$$\widehat{t^*} := \underset{1\leqslant k<n}{\arg\min}\left[\min_{\vartheta_1,\vartheta_2}\left\{\sum_{t=1}^{k}\left(W_t^2 - \vartheta_1^2\right)^2 + \sum_{t=k+1}^{n}\left(W_t^2 - \vartheta_2^2\right)^2\right\}\right], \qquad (3)$$

where $W_t = (X_t - m(\rho; Z_{t-1})) / \delta_0 (Z_{t-1})$. Thus, the change location is estimated by minimizing the sum of squares of residuals among all possible sample splits. Letting

$$\overline{W}_k = \frac{1}{k} \sum_{t=1}^{k} W_t^2, \; \overline{W}_{n-k} = \frac{1}{n-k} \sum_{t=k+1}^{n} W_t^2 \text{ and } \overline{W} = \frac{1}{n} \sum_{t=1}^{n} W_t^2,$$

it is easily seen that for some k, the LS estimator of $\vartheta_1^2 \, (t \leqslant k)$ and $\vartheta_2^2 \, (t > k)$ are $\overline{W}_k$ and $\overline{W}_{n-k}$, respectively, and that (3) can be written as

$$\widehat{t}^* = \underset{1 \leqslant k < n}{\arg\min} \left\{ \sum_{t=1}^{k} \left(W_t^2 - \overline{W}_k \right)^2 + \sum_{t=k+1}^{n} \left(W_t^2 - \overline{W}_{n-k} \right)^2 \right\} = \underset{1 \leqslant k < n}{\arg\min} \, S_k^2. \tag{4}$$

Let $S^2 = \sum_{t=1}^{n} \left(W_t^2 - \overline{W} \right)^2$. A simple algebra gives

$$S^2 = S_k^2 + U_k, \tag{5}$$

where

$$U_k = k \left(\overline{W}_k - \overline{W} \right)^2 + (n-k) \left(\overline{W}_{n-k} - \overline{W} \right)^2. \tag{6}$$

From (4) and (5), we have

$$\widehat{t}^* = \underset{1 \leqslant k < n}{\arg\min} \, \left(S^2 - U_k \right) = \underset{1 \leqslant k < n}{\arg\max} \, U_k. \tag{7}$$

From (6), a simple algebraic computation gives the following alternative expression for U_k :

$$U_k = \frac{n}{k(n-k)} \left(\sum_{t=1}^{k} \left(W_t^2 - \overline{W} \right) \right)^2 = \left(\sqrt{\frac{n}{k(n-k)}} \sum_{t=1}^{k} \left(W_t^2 - \overline{W} \right) \right)^2 = T_k^2. \tag{8}$$

It results from (7) and (8) that

$$\widehat{t}^* = \underset{1 \leqslant k < n}{\arg\max} \, T_k^2 = \underset{1 \leqslant k < n}{\arg\max} \, |T_k|. \tag{9}$$

Writing $T_k^2 = n \Delta_k^2$, it is immediate that

$$\Delta_k^2 = \frac{1}{k(n-k)} \left(\sum_{t=1}^{k} \left(W_t^2 - \overline{W} \right) \right)^2 = \frac{k}{(n-k)} \left(\overline{W}_k - \overline{W} \right)^2.$$

Simple computations give

$$\Delta_k^2 = \frac{k(n-k)}{n^2} \left(\overline{W}_{n-k} - \overline{W}_k \right)^2,$$

from which we have

$$\widehat{t}^* = \underset{1 \leqslant k < n}{\arg\max} \, \Delta_k^2 = \underset{1 \leqslant k < n}{\arg\max} \, \left| \Delta_k \right|. \tag{10}$$

The test statistic we use for testing H_0 against H_1 is a scale version of $\max\limits_{1 \leqslant k \leqslant n-1} |T_k|$.

One can observe that under some conditions (e.g., ε_t i.i.d. with $\varepsilon_t \sim \mathcal{N}(0,1)$), this statistic is the equivalent likelihood based test statistic for testing H_0 against H_1 (see, e.g., Hawkins (1977)). Let

$$C_k = \sum_{t=1}^{k} W_t^2, \quad C_{n-k} = \sum_{t=k+1}^{n} W_t^2 \quad \text{and} \quad C_n = \sum_{t=1}^{n} W_t^2. \tag{11}$$

By simple calculations, we obtain

$$T_k = \sqrt{\frac{n}{k(n-k)}} \sum_{t=1}^{k} \left(W_t^2 - \overline{W}\right) = \left(q\left(\frac{k}{n}\right)\right)^{-1} \left(\frac{1}{\sqrt{n}}\left(C_k - \frac{k}{n}C_n\right)\right), \tag{12}$$

where $q(\cdot)$ is a positive weight function defined for any $x \in (0,1)$ by $q(x) = \sqrt{x(1-x)}$.

3.2 Asymptotic Distribution of the Test Statistic

The study of the asymptotic distribution of the test statistic under H_0, is based on that of the process $\xi_n(\cdot)$ defined for any $s \in [0,1]$ by

$$\xi_n(s) = C_n(s) - sC_n(1), \tag{13}$$

where

$$C_n(s) = \begin{cases} 0 & \text{if } 0 \leqslant s < \dfrac{1}{n} \text{ and } 1 - \dfrac{1}{n} < s < 1 \\ \displaystyle\sum_{t=1}^{[ns]} W_t^2 & \text{if } \dfrac{1}{n} \leqslant s \leqslant 1 - \dfrac{1}{n} \\ \displaystyle\sum_{t=1}^{n} W_t^2 & \text{if } s = 1, \end{cases} \tag{14}$$

where we recall that $[ns]$ is the integer part of ns. For some $\delta \in (1/n, 1/2)$ and for any s in $[\delta, 1-\delta]$, we define

$$T_n(s) = \frac{\xi_n(s)}{\sqrt{n}q(s)} \quad \text{and} \quad \Lambda_n = \max_{\delta \leqslant s \leqslant 1-\delta} \frac{|T_n(s)|}{\widehat{\sigma}_w}, \tag{15}$$

where $q(s) = \sqrt{s(1-s)}$ and $\widehat{\sigma}_w$ is any consistent estimator of

$$\sigma_w^2 = \mathbb{E}\left(W_1^2 - \mathbb{E}\left(W_1^2\right)\right)^2 + 2\sum_{t \geqslant 2} \mathbb{E}\left(\left(W_1^2 - \mathbb{E}\left(W_1^2\right)\right)\left(W_t^2 - \mathbb{E}\left(W_t^2\right)\right)\right).$$

For $\delta \in (0, 1/2)$, we denote by $D_\delta \equiv D([\delta, 1-\delta])$ the space of all right continuous functions with left limits on $[\delta, 1-\delta]$ endowed with the Skorohod metric. It is clear that $C_n(\cdot), \xi_n(\cdot) \in D_0$ and $T_n(\cdot) \in D_\delta$.

Theorem 2 *Assume that the assumptions (A_1)-(A_4) hold. Then, under H_0, we have*

$$\frac{\xi_n(s)}{\sigma_w\sqrt{n}} \xrightarrow{d} \widetilde{B}(s) \text{ in } D_0 \text{ as } n \longrightarrow \infty \quad \text{and} \quad \Lambda_n \xrightarrow{\mathcal{D}} \sup_{\delta \leqslant s \leqslant 1-\delta} \frac{|\widetilde{B}(s)|}{q(s)} \text{ as } n \longrightarrow \infty,$$

where $\left\{\widetilde{B}(s),\ 0 \leqslant s \leqslant 1\right\}$ is a Brownian Bridge on $[0,1]$.

It is worth noting that if the change occurs at the very beginning or at the very end of the data, we may not have sufficient observations to obtain consistent LSE estimators of the parameters or these may not be unique. This is why we stress on the truncated version of the test statistic given in Zou et al. (2007) that we recall :

$$\Lambda_n = \max_{\frac{\nu}{n} \leqslant s \leqslant 1-\frac{\nu}{n}} \frac{|T_n(s)|}{\widehat{\sigma}_w}, \quad \text{for any} \quad 1 \leqslant \nu < \frac{n}{2}.$$

By Theorem 2, it is easy to see that for any $1 \leqslant \nu < n/2$,

$$\sup_{\frac{\nu}{n} \leqslant s \leqslant 1-\frac{\nu}{n}} \frac{|T_n(s)|}{\widehat{\sigma}_w} - \sup_{\frac{\nu}{n} \leqslant s \leqslant 1-\frac{\nu}{n}} \frac{|\widetilde{B}(s)|}{q(s)} \xrightarrow{\mathcal{D}} 0 \text{ as } n \longrightarrow \infty,$$

which yields the asymptotic null distribution of the test statistic. With this, at level of significance $\alpha \in (0,1)$, H_0 is rejected if $\Lambda_n > C_{\alpha,n}$, where $C_{\alpha,n}$ is the $(1-\alpha)$-quantile of the distribution of the above limit. This quantile can be computed by observing that under H_0, for larger values of n, one has

$$\alpha = \mathbb{P}\Bigg(\sup_{\frac{\nu}{n} \leqslant s \leqslant 1-\frac{\nu}{n}} \frac{|T_n(s)|}{\widehat{\sigma}_w} > C_{\alpha,n}\Bigg) \approx \mathbb{P}\Bigg(\sup_{h_\nu(n) \leqslant s \leqslant 1-h_\nu(n)} \frac{|\widetilde{B}(s)|}{q(s)} > C_{\alpha,n}\Bigg), \quad \text{where} \quad h_\nu(n) = \frac{\nu}{n}.$$

From the following relation (1.3.26) of Csörgő and Horváth (1997), for each $h_\nu(n) > 0$, and for larger real number x, we have

$$\begin{aligned} \mathbb{P}\Bigg\{\sup_{h_\nu(n) \leqslant s \leqslant 1-h_\nu(n)} \frac{|\widetilde{B}(s)|}{q(s)} \geqslant x\Bigg\} \quad = \quad & \frac{1}{\sqrt{2\pi}} x \exp\left(\frac{-x^2}{2}\right) \left[\ln\left(\frac{(1-h_\nu(n))^2}{h_\nu^2(n)}\right)\right. \\ & \left. -\frac{1}{x^2} \ln\left(\frac{(1-h_\nu(n))^2}{h_\nu^2(n)}\right) + \frac{4}{x^2} + \mathcal{O}\Big(\frac{1}{x^4}\Big)\right], \end{aligned} \tag{16}$$

which gives an approximation of the tail distribution of $\sup_{h_\nu(n) \leqslant s \leqslant 1-h_\nu(n)} |\widetilde{B}(s)|/q(s)$. Thus, using $\widehat{\sigma}_w$, an estimation of $C_{\alpha,n}$ can be obtained from this approximation. Monte Carlo simulations are often carried out to obtain accurate approximations of $C_{\alpha,n}$. In this purpose, it is necessary to make a good choice of ν. We selected $\nu = 0.9 \times n^{4/5}$ as our option, which we found to be a suitable choice for all the cases we examined. But, to avoid the difficulties associated with the computation of $C_{\alpha,n}$, a decision can also be taken by using the p-value method as in Ngatchou-Wandji et al. (2022). That is using the approximation (16), reject H_0 if

$$\mathbb{P}\big(|\widetilde{B}(s)|q(s) > \Lambda_n\big) \leq \alpha.$$

This idea is used in the simulation.

3.3 Rate of Convergence of the Change Location Estimator

For the study of the estimator $\widehat{t^*}$, we let $\kappa = \kappa_n = \vartheta_2^2 - \vartheta_1^2$ and assume without loss of generality that $\kappa_n > 0$ $(\vartheta_2 > \vartheta_1)$, $\kappa_n \longrightarrow 0$ as $n \longrightarrow \infty$ and that the unknown change point t^* depends on the sample size n. We have the following result :

Theorem 3 *Assume that (A_4) is satisfied, $t^*/n \in (a, 1-a)$ for some $0 < a < 1/2$, $t^* = [n\tau]$ for some $\tau \in (0,1)$ and as $n \longrightarrow \infty$, $\kappa_n \longrightarrow 0$ and $\dfrac{\kappa_n \sqrt{n}}{\sqrt{\ln n}} \longrightarrow \infty$. Then, we have*

$$\widehat{t^*} - t^* = O_{\mathbb{P}}\Big(\frac{1}{\kappa_n^2}\Big),$$

3.4 Limit Distribution of the Location Estimator

In this section, we study the asymptotic behavior of the location estimator. We make the additional assumptions that $\kappa_n >> n^{-\frac{1}{2}}$ and that as $n \longrightarrow \infty$,

$$\frac{\kappa_n \sqrt{n}}{\sqrt{\ln n}} \longrightarrow \infty \text{ and } n^{\frac{1}{2}-\zeta}\kappa_n \longrightarrow \infty \text{ for some } \zeta \in \Big(0, \frac{1}{2}\Big).$$

By (10), we have

$$\widehat{t^*} \quad = \quad \underset{1 \leqslant k < n}{\arg\max}\, n \left(\Delta_k^2 - \Delta_{t^*}^2\right). \tag{17}$$

To derive the limiting distribution of $\widehat{t^*}$, we study the behavior of $n\left(\Delta_k^2 - \Delta_{t^*}^2\right)$ for those ks in the neighborhood of t^* such that $k = [t^* + r\kappa_n^{-2}]$, where r varies in an arbitrary bounded interval $[-N, N]$. For this purpose, we define

$$P_n(r) := n\left\{\Delta_n^2\left(\left[t^* + r\kappa_n^{-2}\right]\right) - \Delta_n^2(t^*)\right\},$$

where $\Delta_n(r) = \Delta_{[r]}$. In addition, we define the two-sided standard Wiener process $\{B^*(r),\ r \in \mathbb{R}\}$ as follows :

$$B^*(r) := \begin{cases} B_1(-r) & \text{if } \ r < 0 \\ B_2(r) & \text{if } \ r \geqslant 0, \end{cases}$$

where $B_i(r),\ i = 1,\ 2$ are two independent standard Wiener processes defined on $[0, \infty)$ with $B_i(0) = 0,\ i = 1,\ 2$.

First, we identify the limit of the process $P_n(r)$ on $|r| \leqslant N$ for every given $N > 0$. We denote by $C([-N, N])$ the space of all continuous functions on $[-N, N]$ endowed with the uniform metric.

Proposition 1 *Assume that (A_4) holds, that $t^* = [n\tau]$ for some $\tau \in (0, 1)$ and that as $n \longrightarrow \infty$, $\kappa_n \longrightarrow 0$ and $\dfrac{\kappa_n\sqrt{n}}{\sqrt{\ln n}} \longrightarrow \infty$. Then, for every $0 < N < \infty$, the process $P_n(r)$ converges weakly in $C([-N, N])$ to the process $P(r) = 2\big\{\sigma_w B^*(r) - \frac{1}{2}|r|\big\}$, where $B^*(\cdot)$ is the two-sided standard Wiener process defined above.*

The above results make it possible to achieve a weak convergence result for $n\left(\Delta_k^2 - \Delta_{t^*}^2\right)$ and then apply the Argmax-Continuous Mapping Theorem (Argmax-CMT). We have :

Theorem 4 *Assume that (A_4) is satisfied, that $t^* = [n\tau]$ for some $\tau \in (0, 1)$ and as $n \longrightarrow \infty$, $\kappa_n \longrightarrow 0$ and $\dfrac{\kappa_n\sqrt{n}}{\sqrt{\ln n}} \longrightarrow \infty$. Then we have $\dfrac{\kappa_n^2\left(\widehat{t^*} - t^*\right)}{\sigma_w^2} \xrightarrow{\mathcal{D}} \mathcal{S}$, where $\mathcal{S} := \arg\max\big\{B^*(u) - \frac{1}{2}|u|,\ u \in \mathbb{R}\big\}$.*

This result yields the asymptotic distribution of the change location estimator. Bhattacharya (1987), Picard (1985) and Yao (1987) investigated the density function of the random variable $\mathcal{S}$ (see Lemma 1.6.3 of Csörgő and Horváth (1987) for more details). They also showed that $\mathcal{S}$ has a symmetric (with respect to 0) probability density function $\gamma(\cdot)$ defined for any $x \in \mathbb{R}$ by $\gamma(x) = \frac{3}{2}\exp(|x|)\Phi\big(\frac{-3}{2}\sqrt{|x|}\big) - \frac{1}{2}\Phi\big(\frac{-1}{2}\sqrt{|x|}\big)$, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal variable. From this result, a confidence interval for the change-point location can be obtained, if one has consistent estimates of κ_n^2 and σ_w^2. With $\widehat{t^*}$, consistent estimates of ϑ_1^2 and ϑ_2^2 are given, respectively, by

$$\widehat{\vartheta}_1^2 = \overline{W}_{\widehat{t^*}} = \frac{1}{\widehat{t^*}} \sum_{t=1}^{\widehat{t^*}} W_t^2 \text{ and } \widehat{\vartheta}_2^2 = \overline{W}_{n-\widehat{t^*}} = \frac{1}{n-\widehat{t^*}} \sum_{t=\widehat{t^*}+1}^{n} W_t^2.$$

Thus, a consistent estimate of κ_n^2 is given by

$$\widehat{\kappa}_n^2 = \frac{1}{n-\widehat{t^*}} \sum_{t=\widehat{t^*}+1}^{n} W_t^2 - \frac{1}{\widehat{t^*}} \sum_{t=1}^{\widehat{t^*}} W_t^2.$$

A consistent estimator of σ_w^2 that we denote by $\widehat{\sigma}_w^2$ can be easily obtained by taking its empirical counterpart. So, at risk $\alpha \in (0,1)$, letting $q_{1-\frac{\alpha}{2}}$ be the quantile of order $1-\frac{\alpha}{2}$ of the distribution of the random variable $\mathcal{S}$, an asymptotic confidence interval for t^* is given by

$$\mathrm{CI} = \widehat{t^*} \pm \left(\left[q_{1-\frac{\alpha}{2}} \frac{\widehat{\sigma}_w^2}{\widehat{\kappa}_n^2} \right] + 1 \right).$$

Remark 1 *In the case that the parameter ρ is unknown, it can be estimated by the CLS method (see Section 2), and be substituted for its estimator in W_t. Indeed, one can easily show that*

$$\frac{1}{k} \sum_{t=1}^{k} W_t^2 = \frac{1}{k} \sum_{t=1}^{k} \widehat{W}_t^2 + o_{\mathbb{P}}(1) \text{ and } \frac{1}{n-k} \sum_{t=k+1}^{n} W_t^2 = \frac{1}{n-k} \sum_{t=k+1}^{n} \widehat{W}_t^2 + o_{\mathbb{P}}(1),$$

where for any $t = 1, \ldots, n$, $\widehat{W}_t = (X_t - m(\widehat{\rho}_n; Z_{t-1})) / \delta_0 (Z_{t-1})$ and $\widehat{\rho}_n$ is the conditional least squares estimators of ρ obtained from Theorem 1. Hence, the same techniques as in the case where ρ is known can be used.

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