KERNEL DENSITY ESTIMATION FOR CONTINUOUS TIME PROCESSES ON RIEMANNIAN MANIFOLD

Djack Guy-Aude Kouadio [1,2] , Vincent Monsan [1] & Anne-Françoise Yao [3,4]

[1] *Laboratoire de Mathématiques Informatique et Mécanique, Université Félix Houphouet Boigny, Abidjan - Côte d'Ivoire,*
monsanv@gmail.com and guyaude.kouadio@ulagunes.com
[2] *Université des Lagunes, Abidjan - Côte d'Ivoire,*
guyaude.kouadio@ulagunes.com
[3] *Laboratoire de Mathématiques Blaise Pascal, CNRS UMR 6620, Campus des Cézeaux, 63171 Aubière Cedex, France,*
anne.yao@uca.fr
[4] *Centre de Mathématiques Appliquées, Ecole polytechnique Route de Saclay 91128 Palaiseau Cedex,*
anne.yao@uca.fr

Résumé. Dans ce travail, nous nous intéresserons à l'estimation de la densité marginale, d'un processus stationnaire à temps continu $(X_t, t \in \mathbb{R})$, où les X_t sont de même loi qu'une variable aléatoire X à valeurs dans une sous variété Riemannienne, $(\mathcal{M}, g)$, de $\mathbb{R}^d$ $(2 \leq d < \infty)$ muni d'une mesure μ_g et $(\mathcal{M}, d_g)$ est complète, où d_g est la métrique induite par g. Désignons par f la densité de X par rapport à μ_g, que nous estimerons par une methode non paramétrique. L'estimateur à noyau proposé généralise d'une part l'estimateur classique pour les processus à temps continu à valeurs dans un espace euclidien aux sous-variétés Riemanniennes (voir par exemple Bosq, D. (1998)), et d'autre part l'estimateur proposé par Pelletier, B. (2005) pour des observations indépendantes et identiquement distribuées (*i.i.d.*) aux processus à temps continu. Lors de notre exposé, apres avoir présenté l'estimateur nous donnerons des résultats de convergence sur le comportement asymptotique de celui-ci obtenus sous conditions de mélange.

Mots-clés. *Estimateur à noyau d'une densité, variétés riemanniennes, conditions de mélange, processus stochastique à temps continu.*

Abstract. This work deals with the estimation of the marginal density of stationary continuous time process $(X_t, t \in \mathbb{R})$, where the X_t's are distributed as a random variable X with values in a Riemannian submanifold, $(\mathcal{M}, g)$, of $\mathbb{R}^d (2 \leq d < \infty)$ endowed with a measure μ_g and $(\mathcal{M}, d_g)$ is complete, where d_g is the metric induced by g. Let f be the density of X with respect to μ_g which we will estimate by a non parametric method. In one hand our estimator is a generalization of the classical kernel estimator for continuous time processes with values in Euclidean space (presented for example in Bosq, D. (1998)) to the Riemannian submanifold. In the other hand, our proposal is also a generalization of Pelletier's estimator for continuous time processes. In this talk, we will give some asymptotical results on the behavior of the estimator under some mixing conditions.

Keywords. *Kernel Density estimation, Riemannian manifolds, Mixing condition, time continuous process.*

1 Introduction

The issue of kernel density estimation has been treated by very large number of papers in the literature when the dataset lives in Euclidean space. But, in many situations the observations belong to a Riemannian manifold, $\mathcal{M}$, as for example in biology (we refer to Mardia and al. (2008) and reference therein), spatial statistics, geology, image analysis (as shown, for example in Pennec, X. (2006). The Riemannian manifold $\mathcal{M}$ can be either unknown (we refer, for example, Aamari, E. and Levrard, C. (2019) or Berenfeld, C. and Hoffmann, M. (2021) or Khardani, S. and Yao, A. F. (2022) and references therein) or known. In this work, we focus in the last case. Namely we are interested in the kernel estimation of the density based on observations on $\mathcal{M}$. The literature in this setting is very limited. When the observations are i.i.d. the interesting work of Pelletier, B. (2005) is the reference. But, we can also refer to Kim, Y. and Park, H. (2013), Henry and al. (2013), Berry, T. and Sauer, T. (2017), Kerkyacharian and al. (2020) and Berenfeld, C. and Hoffmann, M. (2021) and references therein.

This work deals with the estimation of the marginal density of stationary time continuous process $(X_t, t \in \mathbb{R})$, where the X_t's are distributed as a random variable X with values in a subriemannian manifold, $(\mathcal{M}, g)$, of $\mathbb{R}^d (2 \leq d < \infty)$ endowed with a measure μ_g and $(\mathcal{M}, d_g)$ is complete, where d_g is the metric induced by g. Let f be the density of X with respect to μ_g. We propose here a kernel estimator of f. In one hand our estimator is a generalization (to the Riemannian manifold setting) of the classical kernel estimator for continuous time processes (presented for example in Bosq, D. (1998)) with values in Euclidean space. In the other hand, our proposal is also a generalization of Pelletier's estimator for continuous time processes.

In this talk, we will give some asymptotical results on the behavior of the estimator under some mixing conditions.

2 General framework

This work concerns any mesurable continuous time stationary process, $(X_t, t \in \mathbb{R})$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values on Riemannian submanifold $(\mathcal{M}, g)$ of $\mathbb{R}^d$ $(2 \leq d < \infty)$ such that the X_t's are distributed as a random variable X with unknown density f on $\mathcal{M}$.

We assume that $(\mathcal{M}, g)$ is endowed with a measure, μ_g and is geodesically complete compact without boundary. Consequently, $(\mathcal{M}, d_g)$ is a complete metric space, where d_g is the metric induced by g. From now, to ease the reading we set $\mu := \mu_g$ and $d(.,.) = d_g(.,.)$. For all

$x \in \mathcal{M}$ we denote by $T_x\mathcal{M}$, the tangent space to $\mathcal{M}$ at x, $\exp_x : T_x\mathcal{M} \to \mathcal{M}$, the exponential map at x, exp_x^{-1} its inverse and $0_x = exp_x^{-1}(x)$. Let $|g_x(v)|^{1/2} = \frac{d\mu}{d\mu_x}(\exp_x(v)) = \frac{d\mu_{\exp_x^* g}}{d\mu_x}(v)$ is the density of $\mu_{\exp_x^* g}$ with respect to μ_x, the Lebesgue measure on $T_x(\mathcal{M})$.

We assume that the injectivity radius of $\mathcal{M}$ is such that, $inj(\mathcal{M}) > 0$ and consider only regular balls in $\mathcal{M}$. We recall that a regular (or convex) ball, $B(x,h)$, is such that $h < h^*$ where $h^* = \min\{inj(\mathcal{M}), \frac{\pi}{2\sqrt{\kappa}}\}$ with κ the supremum of sectional curvatures of $\mathcal{M}$ (see for example [?] for definition) if this upper bound is positive, and $\kappa = 0$ otherwise. Then, $B(x,h) = \exp_x B(h)$, where $B(h)$ is the ball centred at x with radius h.

Our aim is to estimate the unknown density f on $\mathcal{M}$ from the data $(X_t, 0 \leq t \leq T)$ with $T > 0$.

2.1 Our kernel density estimator

We propose the kernel estimator of f such that for all $x \in \mathcal{M}$:

$$f_T(x) = \frac{1}{T}\int_0^T \frac{1}{h_T^d} \frac{K\left(\frac{d(x,X_t)}{h_T}\right)}{|g_x(\exp_x^{-1}(X_t))|^{1/2}} dt, \tag{1}$$

where the kernel K is a function : $\mathbb{R}_+ \longrightarrow \mathbb{R}$ and the bandwidth $h_T < h^*$ and $\lim_{T\to\infty} h_T = 0(+)$.

We study the asymptotic behavior of f_T under the following assumptions.

Assumptions

The dependence of the process will be measured by means of α-mixing. Then, we consider the α-mixing coefficients of the process $(X_t, t \in \mathbb{R})$ is defined by :

$$\alpha(u) = \sup_{t\in\mathbb{R}} \sup_{A\in\sigma(X_s, s\leq t),\, B\in\sigma(X_s, s\geq t+u)} \{|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)|\}, u \geq 0.$$

HK:

1. K satisfies a Lipschitz condition,
2. $supp\ K = [0;1]$,
3. $\int K(\|v\|)dv = 1$,
4. $\int vK(\|v\|)dv = 0_x$ or at least the vector, $\int vK(\|v\|)\, dv$, is orthogonal to $\text{span}\{\text{grad} f(x)\}$
5. $\int \|v\|^2 K(\|v\|)dv < +\infty$.

$\mathbf{Hh_T}$: $Th_T^d \longrightarrow +\infty$, $h_T \longrightarrow 0$ and $h_T < \frac{h^*}{2}$, when $T \to +\infty$

Assume that $h_T < \frac{h^*}{2}$, to ensure that x is locally a central point when using the kernel estimator

Hf : The probability density $f(.)$ is bounded, twice continuously differentiable at any $x \in \mathcal{M}$

H(Γ,q) : The exists $\Gamma \in \mathcal{B}(\mathbb{R}^2)$ containing $D = \{(s,t) \in \mathbb{R}^2 : s = t\}$ and $q \in]2; +\infty]$ such that :

(1) $\tilde{f}_{s,t}$ and $\tilde{f}_{s,t} * exp_x$ exists for $(s,t) \notin \Gamma$;

(2) $\delta_{x,q}(\Gamma) = \sup_{(s,t)\notin\Gamma} \|\tilde{f}_{s,t} * exp_x\|_{L^q(\mathbb{R}^{2d})} < +\infty$;

(3) there exists a positive constant l_Γ such that

$$l_\Gamma = \limsup_{T\to+\infty} \frac{1}{T} \int_{[0,T]^2\cap\Gamma} dsdt,$$

where $f_{(X_s,X_t)}$ denote the joint probability density of (X_t) and

$$\tilde{f}_{s,t} = f_{(X_s,X_t)} - f \otimes f, \quad \forall (s,t) \in [0,T]^2$$
$$\tilde{f}_{s,t} * exp_x := \tilde{f}_{s,t}\left(exp_x(\cdot), exp_x(\cdot)\right), \quad \forall (s,t) \in [0,T]^2,\ \forall x \in \mathcal{M}.$$

$\mathbf{M}(\gamma, \beta)$ **:** The mixing coefficients of the process (X_t) satisfies $\alpha(|s-t|) = \gamma|s-t|^{-\beta}$ for $(s,t) \notin \Gamma$, where $\gamma > 0$ and $\beta > 0$.

3 Consistency results

3.1 Rates of convergence in Mean Squared Error(MSE)

Theorem 3.1. Under assumptions **HK**, $\mathbf{Hh_T}$, **Hf**, $\mathbf{H(\Gamma, q)}$ and $\mathbf{M}(\gamma,\beta)$ hold for Γ, q, γ and $\beta \geq \max(\frac{2(q-1)}{q-2}, 1)$.

$$MSE(x) := \mathbb{E}\left(f_T(x) - f(x)\right)^2 = O\left(h_T^4 + \frac{1}{Th_T^d}\right) ; \tag{2}$$

Furthermore since f and K are bounded, and $\mathcal{M}$ is compact, then

$$\sup_{x\in\mathcal{M}} MSE(x) = O\left(h_T^4 + \frac{1}{Th_T^d}\right). \tag{3}$$

3.2 Rates of convergence in Mean Integrated Squared Error(MISE)

Theorem 3.2. Under assumptions $\mathbf{HK}$, $\mathbf{Hh_T}$, $\mathbf{Hf}$, $\mathbf{H(\Gamma, q)}$ and $\mathbf{M}(\gamma, \beta)$ hold for Γ, q, γ and $\beta \geq \max(\frac{2(q-1)}{q-2}, 1)$.

Since f and K are bounded, and $\mathcal{M}$ is compact, we have

$$MISE := \int_{\mathcal{M}} MSE(x)dx = O\left(h_T^4 + \frac{1}{Th_T^d}\right). \tag{4}$$

3.3 Convergence in probability

Theorem 3.3. Under the assumptions $\mathbf{HK}$, $\mathbf{Hf}$, $\mathbf{Hh_T}$, $\mathbf{H(\Gamma, q)}$ and $\mathbf{M}(\gamma, \beta)$ hold for Γ, q, γ and $\beta \geq \max\left(\frac{2(q-1)}{q-2}, 1\right)$.

If $Th_T^{4+d}(\log T)^{-1} \to 0$ and $Th_T^{\frac{d(\beta+1)}{\beta-1}}(\log T)^{-1} \to \infty$, then

$$f_T(x) - f(x) = \mathcal{O}_p\left(\left(\frac{\log T}{Th_T^d}\right)^{1/2}\right). \tag{5}$$

3.4 Almost sure convergence

Now to get the almost sure consistency rate result, we apply the Borel-Cantelli Lemma 4.2 of Bosq, D. (1998) to the sample paths of $Q_T = \left(\frac{\log T}{Th_T^d}\right)^{1/2}(f_T(x) - \mathbb{E}(f_T(x)))$.

Assume that the paths of $Q_T = \left(\frac{\log T}{Th_T^d}\right)^{1/2}(f_T(x) - \mathbb{E}(f_T(x)))$ are uniformly continuous with probability 1 and the bandwidth h_T is such that :

$$\varphi_\eta(T) = C\left(T^{-1}h_T^{-\frac{d(\beta+1)}{\beta-1}}\log T\right)^{\frac{\beta-1}{2}} \quad \text{is decreasing} \tag{6}$$

$$Th_T^{\frac{d(\beta+1)}{\beta-3}}(\log T)^{-\frac{\beta-1}{\beta-3}}g(T)^{\frac{-2}{\beta-3}} \to \infty \tag{7}$$

where $g(T) = \log(T)\left(\log(\log T)\right)^{1+\epsilon}$, and $\epsilon > 0$ is an arbitrary small real number. Then we get the following strong consistency result.

Theorem 3.4. Under the assumptions (6), (7), $\mathbf{HK}$, $\mathbf{Hf}$, $\mathbf{Hh_T}$, $\mathbf{H(\Gamma, q)}$ and $\mathbf{M}(\gamma, \beta)$ hold for Γ, q, γ and $\beta \geq \max\left(\frac{2(q-1)}{q-2}, 3\right)$. If $Th_T^{4+d}(\log T)^{-1} \to 0$, then

$$f_T(x) - f(x) = \mathcal{O}_{a.s.}\left(\left(\frac{\log T}{Th_T^d}\right)^{1/2}\right). \tag{8}$$

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References

Aamari, E. and Levrard, C. (2019), Non-asymptotic rates for manifold, tangent space and-curvature estimation, *The Annals of Statistics*, 47(1):177 – 204.

Berenfeld, C. and Hoffmann, M. (2021), Density estimation on an unknown submanifold, *Electronic Journal of Statistics*, 15(1):2179 – 2223

Berry, T. and Sauer, T. (2017), Density estimation on manifolds with boundary, *Computational Statistics & Data Analysis*, 107:1–17.

Bosq, D. (1998), *Nonparametric statistics for stochastic processes , estimation and prediction.Second Edition*, Springer.

Cleanthous, G., Georgiadis, A. G., Kerkyacharian, G., Petrushev, P., and Picard, D. (2020), Kernel and wavelet density estimators on manifolds and more general metric spaces, *Bernoulli*, 26(3):1832 – 1862.

Gallot, S., Hulin, D., and Lafontaine, J. (2004), Riemmanian geometry .

Henry, G., Munoz, A., and Rodriguez, D. (2013), Locally adaptive density estimation on riemannian manifolds, *SORT-Statistics and Operations Research Transactions*, pages 111–130.

Khardani, S. and Yao, A. F. (2022), Nonparametric recursive regression estimation on riemannian manifolds, *Statistics & Probability Letters*, 182:109274.

Kim, Y. and Park, H. (2013), Geometric structures arising from kernel density estimation on riemannian manifolds, *Journal of Multivariate Analysis*, 114:112–126.

Mardia, K. V., Hughes, G., Taylor, C. C., and Singh, H. (2008), A multivariate von misesdistribution with applications to bioinformatics, *Canadian journal of statistics*, 36(1):99–109.

Pelletier, B. (2005), Kernel density estimation on riemannian manifolds, *Statistics & probability letters*, 73(3):297–304.

Pennec, X. (2006), Intrinsic statistics on riemannian manifolds: Basic tools for geometric measurements, *Journal of mathematical imaging and vision*, 25(1):127–154.